

# ON THE LAW OF LARGE NUMBERS FOR NONMEASURABLE IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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**ABSTRACT.** Let  $\Omega$  be a countable infinite product  $\Omega^{\mathbb{N}}$  of copies of the same probability space  $\Omega_1$ , and let  $\{\Xi_n\}$  be the sequence of the coordinate projection functions from  $\Omega$  to  $\Omega_1$ . Let  $\Psi$  be a possibly non-measurable function from  $\Omega_1$  to  $\mathbb{R}$ , and let  $X_n(\omega) = \Psi(\Xi_n(\omega))$ . Then we can think of  $\{X_n\}$  as a sequence of independent but possibly non-measurable random variables on  $\Omega$ . Let  $S_n = X_1 + \cdots + X_n$ . By the ordinary Strong Law of Large Numbers, we almost surely have  $E_*[X_1] \leq \liminf S_n/n \leq \limsup S_n/n \leq E^*[X_1]$ , where  $E_*$  and  $E^*$  are the lower and upper expectations. We ask if anything more precise can be said about the limit points of  $S_n/n$  in the non-trivial case where  $E_*[X_1] < E^*[X_1]$ , and obtain several negative answers. For instance, the set of points of  $\Omega$  where  $S_n/n$  converges is maximally nonmeasurable: it has inner measure zero and outer measure one.

## 1. INTRODUCTION

Ordinary random variables are  $P$ -measurable functions on a probability space  $(\Omega, \mathcal{F}, P)$ , where  $\mathcal{F}$  is a  $\sigma$ -field on  $\Omega$ . By the ordinary Strong and Weak Laws of Large Numbers (LLNs), if  $X_1, X_2, \dots$  are measurable identically distributed random variables with finite expectation, then  $(X_1 + \cdots + X_n)/n \rightarrow E[X_1]$  almost surely (Strong Law) and in probability (Weak Law). But we can also ask what happens to long-term means of samples when the random variables are not measurable.

Let  $(\Omega, \mathcal{F}, P)$  be a probability space. The following collects some known facts (see, e.g., [5, Lemmas 1.2.2 and 1.2.3]) that allow us to apply probabilistic techniques in the case of nonmeasurable random variables.

**Proposition 1.** *Let  $H$  be any subset of  $\Omega$ . Then there are measurable sets  $H_*$  and  $H^*$  such that  $H_* \subseteq H \subseteq H^*$  and such that for any measurable  $A \subseteq H$  we have  $P(A) \leq P(H_*)$  and for any measurable  $B \supseteq H$  we have  $P(B) \geq P(H^*)$ . The sets  $H_*$  and  $H^*$  are uniquely defined up to sets of measure zero.*

*For any real-valued function  $f$  on  $\Omega$ , there are measurable functions  $f_*$  and  $f^*$  such that  $f_* \leq f \leq f^*$  everywhere and for any measurable  $g$  on  $\Omega$  such that  $g \leq f$  everywhere, we have  $g \leq f_*$  almost surely, while for any measurable  $h$  on  $\Omega$  such that  $f \leq h$  everywhere, we have  $h \geq f^*$  almost*

surely. The functions  $f_*$  and  $f^*$  are uniquely defined up to almost sure equality.

The functions  $f_*$  and  $f^*$  are the *maximal measurable minorant* and *minimal measurable majorant* of  $f$ , respectively.

We then have  $P_*(H) = P(H_*)$  and  $P^*(H) = P(H^*)$ , where  $P_*$  and  $P^*$  are the inner and outer measures generated by  $P$ . Note that  $H$  is measurable with respect to the completion of  $P$  if and only if  $P_*(H) = P^*(H)$ , in which case it has that value as its measure with respect to the completion of  $P$ . If, as some confirmation theorists are wont to do (e.g., [3]), we were to deal in interval-valued probabilities, we might reasonably define the probability of  $H$  as  $[P_*(H), P^*(H)]$ .

We say that a set is *maximally nonmeasurable* provided that  $P_*(H) = 0$  and  $P^*(H) = 1$ . Such a set is one all of whose measurable subsets have null measure and all of whose measurable supersets have full measure.

As a replacement for the independence assumption in the case of ordinary random variables, take our probability space  $(\Omega, \mathcal{F}, P)$  to be a product of the probability spaces  $(\Omega_n, \mathcal{F}_n, P_n)$ , and let our sequence of possibly nonmeasurable random variables be a sequence of functions  $X_1, X_2, \dots$  on  $\Omega$  such that  $X_n(\omega_1, \omega_2, \dots)$  depends only on the value of  $\omega_n$ , so that there is a function  $\Psi_n$  such that  $X_n(\omega_1, \omega_2, \dots) = \Psi_n(\omega_n)$ . We will say that  $X_1, X_2, \dots$  is then a sequence of independent identically-distributed possibly-nonmeasurable random variables (iidpnmrvs) providing that all the probability spaces  $(\Omega_n, \mathcal{F}_n, P_n)$  are the same space  $(\Omega_1, \mathcal{F}_1, P_1)$  and that  $\Psi_n$  is the same function  $\Psi_1$  for all  $n$ .

The following fact about product measures and the  $(\cdot)_*$  and  $(\cdot)^*$  operators follows from [5, Lemma 1.2.5].

**Proposition 2.** *Suppose  $(\Omega, \mathcal{F}, P)$  is a product of the probability spaces  $(\Omega_n, \mathcal{F}_n, P_n)$  for  $n = 1, 2, \dots$ . Let  $\Psi_n$  be a function on  $\Omega_n$ . Let  $X_n(\omega_1, \omega_2, \dots) = \Psi_n(\omega_n)$ . Let  $Y_n(\omega_1, \omega_2, \dots) = (\Psi_n)_*(\omega_n)$  and  $Z_n(\omega_1, \omega_2, \dots) = \Psi_n^*(\omega_n)$ . Then  $P$ -almost surely we have  $(X_n)_* = Y_n$  and  $X_n^* = Z_n$ .*

In particular, if  $X_1, \dots, X_n$  are iidpnmrvs, then  $(X_1)_*, \dots, (X_n)_*$  are identically distributed independent random variables, and so are  $X_1^*, \dots, X_n^*$ . Let  $S_n = X_1 + \dots + X_n$ . From the Strong Law of Large Numbers as applied to  $\{(X_n)_*\}$  and  $\{X_n^*\}$  (and using the fact that if  $|X_1|^*$  is integrable, then so are  $(X_1)_*$  and  $X_1^*$ ) it then follows that almost surely:

$$(1) \quad E[(X_1)_*] \leq \liminf_{n \rightarrow \infty} \frac{S_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{S_n}{n} \leq E[X_1^*].$$

Here and elsewhere “almost surely” will mean *except perhaps on a set of probability zero*. Thus an event holds almost surely provided its lower probability is 1. (In the case of complete measures, this is equivalent to the usual notion of holding almost surely as holding on a set of full measure.)

We can define the *lower* and *upper expectations* of  $X_1$  as  $E_*[X_1] = E[(X_1)_*]$  and  $E^*[X_1] = E[X_1^*]$ , respectively (for more on lower and upper expectations, see [2]). Again, we have a trivial case when  $E[(X_1)_*] = E[X_1^*]$  and then the Strong Law holds. In that case,  $(X_n)_* = X_n^*$  almost surely (since  $(X_n)_* \leq X_n \leq X_n^*$ ), and  $X_n$  will be measurable with respect to the completion of  $P$ .

We can now ask whether (1) can be improved on in any way. For instance, can the first or last almost sure inequality sometimes be replaced by an equality? Or can we say that in the non-trivial case it is almost surely true that  $S_n/n$  diverges? Our main result shows that the answers to these questions are negative.

**Theorem 1.** *Let  $X_1, X_2, \dots$  be iidpnmrsvs with  $E^*[|X_1|] < \infty$ . Suppose  $A$  is a non-empty proper subset of  $[E_*[X_1], E^*[X_1]]$ . Then each of the following is maximally nonmeasurable:*

- (i) *the subset of  $\Omega$  where  $\liminf_{n \rightarrow \infty} S_n(n)/n$  is in  $A$*
- (ii) *the subset of  $\Omega$  where  $\limsup_{n \rightarrow \infty} S_n(n)/n$  is in  $A$*
- (iii) *the subset of  $\Omega$  where  $\lim_{n \rightarrow \infty} S_n(n)/n$  exists and is in  $A$*
- (iv) *the subset of  $\Omega$  where  $\lim_{n \rightarrow \infty} S_n(n)/n$  exists*
- (v) *the subset of  $\Omega$  where all the limit points of  $S_n(n)/n$  are in  $A$ .*

Thus in the non-trivial case ( $[E_*[X_1], E^*[X_1]]$  has a non-empty proper subset if and only if  $E_*[X_1] < E^*[X_1]$ ) nothing can be probabilistically said, with respect to  $P$ , about the limit points of  $S_n(n)/n$  except that all the limit points lie within  $[E_*[X_1], E^*[X_1]]$ .

For completeness, here is a somewhat analogous result about the Weak Law:

**Theorem 2.** *Let  $X_1, X_2, \dots$  be iidpnmrsvs with  $E^*[|X_1|] < \infty$ . Suppose  $a \in [E[(X_1)_*], E[X_1^*]]$  and that  $\varepsilon > 0$  is sufficiently small that  $[E_*[X_1], E^*[X_1]]$  is not a subset of  $[a-\varepsilon, a+\varepsilon]$ . Then  $P_*(|S_n/n - a| > \varepsilon) \rightarrow 0$  and  $P^*(|S_n/n - a| > \varepsilon) \rightarrow 1$  as  $n \rightarrow \infty$ .*

The proof of both theorems will be based on the following fact about the existence of extensions of measures.

**Lemma 1.** *Suppose  $f$  is a function on a probability space  $(\Omega, \mathcal{F}, P)$  and  $f$  is simple, i.e., takes on only finitely many values. Then there are extensions  $P_*$  and  $P^*$  of  $P$  defined on the  $\sigma$ -field generated by  $\mathcal{F}$  and  $f$  such that  $f = f_*$  almost surely with respect to  $P_*$  and  $f = f^*$  almost surely with respect to  $P^*$ .*

*Remark.* Our proofs of the theorems would be much simpler if we could have this for  $f$  taking on infinitely many values, but alas the lemma is false in that case. To see this falsity, let  $\Omega$  be the open square  $(0, 1)^2$ , make  $\mathcal{F}$  be the  $\sigma$ -field of subsets of the form  $A \times (0, 1)$  for  $A \subseteq (0, 1)$  Lebesgue-measurable, take  $P$  to be the restriction of Lebesgue measure to  $\mathcal{F}$ , and set  $f(x, y) = y$ . Then  $f_* = 0$  almost surely with respect to  $P$  but there is no extension of  $P$  with respect to which  $f = 0$  almost surely, since  $f$  is nowhere equal to zero.

## 2. PROOFS

The following simple fact will be useful. As usual,  $U \Delta V = (U - V) \cup (V - U)$ .

**Lemma 2.** *Suppose  $B$  is such that  $P_*(B) = 0$ . If  $U$  and  $V$  are measurable sets such that  $U - B = V - B$ , then  $U \Delta V$  is a null set.*

*Proof.* If  $U - B = V - B$ , then  $U - V \subseteq B$  and  $V - U \subseteq B$ . But if  $P_*(B) = 0$ , then all measurable subsets of  $B$  are null sets.  $\square$

*Proof of Lemma 1.* We shall prove the result about  $f_*$ ; the rest of the lemma follows by applying this to  $-f$ .

We need only prove our result in the special case where  $f \geq 0$  everywhere and  $f_* = 0$  almost surely. For suppose we have proved this special case and are given a general simple  $f$ . It is easy to check that if  $f$  is simple, we can choose  $f_*$  to be simple, e.g., by setting  $f_*(\omega) = \max\{y : \omega \in \{\omega' : f(\omega') \geq y\}_*\}$ . Using the easy fact that for any measurable  $g$  we have  $(f - g)_* = f_* - g$  almost surely, we have  $(f - f_*)_* = 0$  almost surely. Let  $h = f - f_*$ . Then  $h \geq 0$  everywhere and  $h_* = 0$  almost surely. By the special case of the lemma, there is an extension  $P_*$  of  $P$  to the  $\sigma$ -field generated by  $\mathcal{F}$  and  $h$  such that  $P_*$ -almost surely  $h = 0$ . Thus  $P_*$ -almost surely  $f_* = g$ . But the  $\sigma$ -field generated by  $\mathcal{F}$  and  $h$  is the same as that generated by  $\mathcal{F}$  and  $f$  since  $f$  and  $h$  differ only by a measurable function, and so our proof will be complete.

So, suppose that  $f \geq 0$  everywhere and  $f_* = 0$  almost surely. Let  $A = \{\omega : f(\omega) > 0\}$ . Observe that  $P_*(A) = 0$ . For suppose that  $B \subseteq A$  is measurable. Let  $y_1$  be the smallest non-zero value that  $f$  takes. Then  $f \geq y_1 \cdot 1_B$  everywhere, and so  $f_* \geq y_1 \cdot 1_B$  almost surely, which implies that  $P(B) = 0$ . Thus the only measurable subsets of  $A$  are null, and so  $P_*(A) = 0$ .

Now, let  $\mathcal{F}_1$  be the collection of all sets of the form  $(U \cap A) \cup (V \cap A^c)$ , for  $U$  and  $V$  in  $\mathcal{F}$ . It is easy to check that this is equal to the  $\sigma$ -field generated by  $\mathcal{F}$  and  $A_1$ .

Define  $P_1((U \cap A) \cup (V \cap A^c)) = P_1(V)$ . First we need to check that this is well-defined. Suppose  $(U \cap A) \cup (V \cap A^c) = (U' \cap A) \cup (V' \cap A^c)$ . Then  $V - A = V \cap A^c = V' \cap A^c = V' - A$ . Since  $P_*(A) = 0$ , it follows from Lemma 2 that  $V \Delta V'$  is a null set. Thus  $P(V) = P(V')$  and so  $P_1$  is well defined.

It is an easy exercise to verify that  $P_1$  is a probability measure that extends  $P$  and satisfies  $P_1(A) = 0$ . Thus  $f = 0$  except on a  $P_1$ -null set. Let  $(\Omega, P_1^*, \mathcal{F}_1^*)$  be the completion of  $(\Omega, P_1, \mathcal{F}_1)$ . Any function which is constant except on a set of measure zero is measurable with respect to the completion of the measure, and hence  $f$  is measurable with respect to  $\mathcal{F}_1^*$ . Letting  $P_*$  be the restriction of  $P_1^*$  to the  $\sigma$ -field generated by  $\mathcal{F}$  and  $f$ , the proof is complete.  $\square$

For the proofs of the theorems we will need this simple consequence of [5, Lemma 1.2.2].

**Lemma 3.** *If  $|f - g| \leq \varepsilon$  everywhere, then  $|f_* - g_*| \leq \varepsilon$  and  $|f^* - g^*| \leq \varepsilon$  almost surely.*

**Lemma 4.** *Suppose that  $f = g$  on a measurable set  $B$ . Then  $f_* = g_*$  and  $f^* = g^*$  almost surely on  $B$ .*

*Proof.* As usual, we only need to show half of the result, say that  $f^* = g^*$  almost surely on  $B$ . Let  $h(\omega) = f^*(\omega)$  for  $\omega \notin B$  and let  $h(\omega) = \min(f^*(\omega), g^*(\omega))$  otherwise. Then  $h$  is a measurable function such that  $h \geq f$ . Hence  $h \geq f^*$  almost surely. Then  $g^* \geq h \geq f^*$  almost surely on  $B$ . In the same way, we see that  $f^* \geq g^*$  almost surely on  $B$ .  $\square$

The following trivial lemma provides the strategy for the proof of our theorems.

**Lemma 5.** *Suppose that  $B$  is subset of a probability space  $(\Omega, \mathcal{F}, P)$  such that there are extensions  $P_1$  and  $P_2$  of  $P$  so that  $B$  is  $P_1$ - and  $P_2$ -measurable with  $P_1(B) = x_1$  and  $P_2(B) = x_2$ . Then  $P_*(B) \leq x_1$  and  $x_2 \leq P^*(B)$ . In particular, if  $P_1(B) = 0$  and  $P_2(B) = 1$ , then  $B$  is maximally nonmeasurable.*

*Proof.* We have  $P_*(B) = P(B_*) = P_1(B_*) \leq P_1(B) = x_1$  and  $P^*(B) = P(B^*) = P_2(B^*) \geq P_2(B) = x_2$ , where  $B_*$  and  $B^*$  are defined with respect to  $P$ .  $\square$

By Lemma 5, we need to show that for each of the subsets of  $\Omega$  mentioned in Theorem 1, there is an extension of  $P$  that assigns measure zero to the subset and another that assigns it measure one.

As we will soon see, the following lemma will yield all the results of Theorem 1.

**Lemma 6.** *Suppose  $X_1, X_2, \dots$  is a sequence of iidpnmrsvs such that  $E[|X_1|^*] < \infty$ . Then for any  $\alpha \in [E_*[X_1], E^*[X_1]]$  there is an extension  $P'$  of  $P$  such that  $P'$ -almost surely  $S_n/n$  converges to  $\alpha$ . If  $E_*[X_1] < E^*[X_1]$  then there is an extension  $P''$  of  $P$  such that  $P''$ -almost surely  $S_n$  diverges.*

*Proof.* Write  $E_P[f]$  for the expectation of  $f$  with respect to  $P$ , i.e.,  $\int_{\Omega} f(\omega) dP(\omega)$ . The variables  $X_1, X_2, \dots$  are defined by  $X_n(\omega_1, \omega_2, \dots) = \Psi_1(\omega_n)$  on our product space  $\Omega$  for some real-valued function  $\Psi_1$ .

Let  $X'_n = X_n \cdot 1_{\{|X_n| \leq n\}}$ .

Let  $A_{n,\pm} = \{\pm X_n > n\}$ . Let  $A_n = \{|X_n| > n\} = A_{n,+} \cup A_{n,-}$ . Observe that  $A_{n,+} \subseteq \{X_n^* > n\}$  and  $A_{n,-} \subseteq \{(X_n)_* < -n\}$ . Let  $B_n = \{|X_n^*| > n\} \cup \{|(X_n)_*| > n\}$ . Clearly  $A_n \subseteq B_n$ . Then:

$$(2) \quad \sum_{n=1}^{\infty} P(B_n) \leq \sum_{n=1}^{\infty} P(|(X_n)_*| > n) + \sum_{n=1}^{\infty} P(|X_n^*| > n) < \infty,$$

since  $\sum_{n=1}^{\infty} P(|X| > n) \leq \int_0^{\infty} P(|X| > t) dt = E[|X|]$ , and since both  $(X_n)_*$  and  $X_n^*$  have finite expectations given that  $|X_n|^*$  does. In particular, almost surely, only finitely many of the  $B_n$  occur by Borel-Cantelli. Since  $A_n \subseteq B_n$ , almost surely only finitely many of the  $A_n$  occur.

I now claim that  $E[(X'_n)_*] \rightarrow E[(X_1)_*]$  and  $E[(X'_n)^*] \rightarrow E[X_1^*]$  as  $n \rightarrow \infty$ . We only need to prove the latter claim since the former follows by applying the latter to the iidpnmr sequence  $\{-X_n\}$ . Now, outside of  $B_n$ , we have  $X'_n = X_n$ , and since  $B_n$  is measurable it follows from Lemma 4 that outside of  $B_n$  we have  $(X'_n)^* = X_n^*$  almost surely. Moreover, everywhere on  $B_n$  we have  $X'_n = 0$  and hence  $(X'_n)^* = 0$  by Lemma 4. Thus:

$$\begin{aligned} |E[(X'_n)^*] - E[X_1^*]| &= |E[(X'_n)^*] - E[X_n^*]| \\ &\leq E[|(X'_n)^* - X_n^*|] \\ &\leq E[|(X'_n)^* - X_n^*| \cdot 1_{B_n}] \\ &= E[|X_n^*| \cdot 1_{B_n}] \\ &\leq E[|X_n^*| \cdot 1_{\{|X_n|^* > n\}}] \\ &= E[|X_1^*| \cdot 1_{\{|X_1|^* > n\}}] \rightarrow 0, \end{aligned}$$

since  $E[|X_1|^*] < \infty$ .

Let  $\Psi'_n = \Psi \cdot 1_{\{|\Psi| \leq n\}}$  so that  $X'_n(\omega_1, \omega_2, \dots) = \Psi'_n(\omega_n)$ . Choose a simple function  $\Psi_{1,n}$  such that both  $|\Psi_{1,n} - \Psi'| \leq 1/n$  and  $|\Psi_{1,n}| \leq n$  everywhere on  $\Omega_1$ .

By Lemma 1 there is an extension  $P_{1,n,0}$  of  $P_1$  such that  $\Psi_{1,n}$  is  $P_{1,n,0}$ -measurable and  $P_{1,n,0}$ -almost surely  $\Psi_{1,n} = (\Psi_{1,n})_*$ , and an extension  $P_{1,n,1}$  of  $P_1$  such that  $\Psi_{1,n,n}$  is  $P_{1,n,1}$ -measurable and  $P_{1,n,1}$ -almost surely  $\Psi_{1,n} = \Psi_{1,n}^*$ . Let  $\mathcal{F}_{1,n}$  be the  $\sigma$ -field on  $\Omega_1$  generated by  $\mathcal{F}_1$  and  $\Psi_{1,n}$ .

For  $i = 1, 2$ , let  $P^i$  be the product of the measures  $P_{1,1,i}, P_{1,2,i}, P_{1,3,i}, \dots$ . This is an extension of  $P$ .

Let  $Y_n(\omega_1, \omega_2, \dots) = \Psi_{1,n}(\omega_n)$ . By Proposition 2,  $(Y_n)_*(\omega_1, \omega_2, \dots) = (\Psi_{1,n})_*(\omega_n)$  for  $P$ -almost all  $(\omega_1, \omega_2, \dots)$ . Then  $P^0$ -almost surely we have  $Y_n = (Y_n)_*$  and  $P^1$ -almost surely  $Y_n = Y_n^*$  (where  $(Y_n)_*$  and  $Y_n^*$  are defined with respect to  $P$ ). Define the measure  $P' = (1-a)P^0 + aP^1$ , where  $a$  is such that  $\alpha = (1-a)E_P[(X_1)_*] + aE_P[X_1^*]$ . Then  $E_{P'}[Y_n] = (1-a)E_P[(Y_n)_*] + aE_P[Y_n^*]$ . Now  $|(Y_n)_* - (X'_n)_*| \leq 1/n$  and  $|Y_n^* - (X'_n)^*| \leq 1/n$  everywhere by Lemma 3 and the choice of  $\Psi_{1,n}$ , so by uniform convergence we have  $E[(Y_n)_* - (X'_n)_*]$  and  $E[Y_n^* - (X'_n)^*]$  converging to zero. Moreover, we have already seen that  $E[(X'_n)_*] \rightarrow E[(X_1)_*]$  and  $E[(X'_n)^*] \rightarrow E[X_1^*]$ , so  $E_P[(Y_n)_*] \rightarrow E_P[(X_1)_*]$  and  $E_P[Y_n^*] \rightarrow E_P[X_1^*]$ . Thus,  $E_{P'}[Y_n] \rightarrow (1-a)E_P[(X_1)_*] + aE_P[X_1^*] = \alpha$ .

Now  $|Y_n - X'_n| \leq 1/n$  everywhere and  $|X'_n| \leq \max(|(X_n)_*|, |X_n^*|) \cdot 1_{B_n}$ . Thus:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\text{Var}_{P'}[Y_n]}{n^2} &\leq \sum_{n=1}^{\infty} \frac{E[Y_n^2]}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{E_{P'}[(X'_n + (Y_n - X'_n))^2]}{n^2} \\
&\leq 2 \sum_{n=1}^{\infty} \frac{E_{P'}[(X'_n)^2]}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^3} \\
&= 2 \sum_{n=1}^{\infty} \frac{E_{P'}[(X_n)_*]^2 \cdot 1_{B_n}}{n^2} + 2 \sum_{n=1}^{\infty} \frac{E_{P'}[(X_n^*)^2 \cdot 1_{B_n}]}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^3} \\
&= 2 \sum_{n=1}^{\infty} \frac{E_P[(X_n)_*]^2 \cdot 1_{B_n}}{n^2} + 2 \sum_{n=1}^{\infty} \frac{E_P[(X_n^*)^2 \cdot 1_{B_n}]}{n^2} + O(1) \\
&\leq 2 \sum_{n=1}^{\infty} \frac{E_P[(X_n)_*]^2 \cdot 1_{\{|(X_n)_*| > n\}}]}{n^2} + 2 \sum_{n=1}^{\infty} \frac{E_P[(X_n^*)^2 \cdot 1_{\{|X_n^*| > n\}}]}{n^2} \\
&\quad + O(1),
\end{aligned}$$

where the last equality follows from the fact that  $P$  is an extension of  $P'$  and  $(X_n)_*$  and  $X_n^*$  are  $P$ -measurable. The finiteness of the right hand side then follows from the proof of [1, Lemma 2.4.3]. It then follows from Kolmogorov's Strong Law [4, Theorem 5.8] that  $P'$ -almost surely  $(Y_1 + \dots + Y_n - E[Y_1 + \dots + Y_n]) \rightarrow 0$  and hence  $(Y_1 + \dots + Y_n)/n \rightarrow \alpha$ .

Let  $B = \bigcup_{n=1}^{\infty} B_n$ . Let  $T_n = Y_1 + \dots + Y_n$ . Let  $S'_n = X'_1 + \dots + X'_n$ . Since  $|X'_n - Y_n| \leq 1/n$  everywhere,  $|S'_n/n - T_n/n| \leq (1/n)(1 + 1/2 + \dots + 1/n)$  and the right hand side converges to zero. Moreover,  $P$ -almost surely for all but finitely many  $n$  we have  $X_n = X'_n$ , so that  $P$ -almost surely  $(S_n - S'_n)/n \rightarrow 0$ . Thus,  $P$ -almost surely we have  $(S_n - T_n)/n \rightarrow 0$ . But  $P'$  extends  $P$ , so this also holds  $P'$ -almost surely. But since  $P'$ -almost surely  $T_n/n \rightarrow \alpha$ , we also have  $S_n/n$  converging  $P'$ -almost surely to  $\alpha$ .

To construct  $P''$ , first recall that  $E_P[(Y_n)_*] \rightarrow E_P[(X_1)_*]$  and  $E_P[Y_n^*] \rightarrow E_P[X_1^*]$ . Let  $\gamma = E_P[(X_1)_*]$  and  $\delta = E_P[X_1^*]$ . For  $n > 0$ , let  $a_n$  be a strictly increasing sequence of positive integers such that (a)  $|E_P[(Y_k)_*] - E_P[(X_1)_*]| < 1/n$  and  $|E_P[Y_k^*] - E_P[X_1^*]| < 1/n$  for all  $k \geq a_n$ , and (b)  $a_{n+1}/a_n \rightarrow \infty$ . Let  $a_0 = 1$ .

Let  $L_n = \{a_{n-1} + 1, a_{n-1} + 1, \dots, a_n\}$ . Let  $L = L_1 \cup L_3 \cup L_5 \cup \dots$ . For  $n \in L$ , let  $\alpha_n = E_P[(Y_n)_*]$  and for  $n \notin L$ , let  $\alpha_n = E_P[Y_n^*]$ . Let  $\beta_n = \alpha_1 + \dots + \alpha_n$ . I now claim that  $\beta_{a_{2n}}/a_{2n} \rightarrow \delta$  and  $\beta_{a_{2n+1}}/a_{2n+1} \rightarrow \gamma$  as  $n \rightarrow \infty$ . For, since both  $E_P[(Y_n)_*]$  and  $E_P[Y_n^*]$  converge, there is an  $M < \infty$  such that

$|\alpha_k| \leq M$  for all  $K$ , and then by the choice of  $a_n$ :

$$\begin{aligned}\beta_{a_{2n}} - a_{2n}\delta &= \sum_{k=a_{2n-1}+1}^{a_{2n}} (E_P[Y_k^*] - \delta) - a_{2n-1}\delta + \sum_{k=1}^{a_{2n-1}} \alpha_k \\ &= O(a_{2n}/(2n-1)) + O(a_{2n-1}).\end{aligned}$$

Since  $a_{2n-1}/a_{2n} \rightarrow 0$  by condition (b) above, we see that  $\beta_{a_{2n}}/a_{2n} - \delta$  converges to 0 as desired. Likewise:

$$\begin{aligned}\beta_{a_{2n+1}} - a_{2n+1}\gamma &= \sum_{k=a_{2n}+1}^{a_{2n+1}} (E_P[(Y_k)_*] - \gamma) - a_{2n}\gamma + \sum_{k=1}^{a_{2n}} \alpha_k \\ &= O(a_{2n+1}/2n) + O(a_{2n}),\end{aligned}$$

and so  $\beta_{a_{2n+1}}/a_{2n+1} \rightarrow \gamma$ .

Now define  $P''_{1,n}$  to be equal to  $P_{1,n,0}$  if  $n \in L$ , and to  $P_{1,n,1}$  if  $n \notin L$ . Let  $P''$  be the product of the measures  $P''_{1,1}, P''_{1,2}, \dots$ .

In exactly the same way as we proved above using Kolmogorov's Strong Law that  $P'$ -almost surely  $(T_n - E_{P'}[T_n])/n \rightarrow 0$ , we can also show that  $P''$ -almost surely  $(T_n - E_{P''}[T_n])/n \rightarrow 0$ . But  $E_{P''}[T_n] = \beta_n$  since  $E_{P''}[Y_n] = E_P[(Y_n)_*]$  if  $n \in L$  and  $E_{P''}[Y_n] = E_P[Y_n^*]$  otherwise. Thus,  $P''$ -almost surely  $T_{a_{2n}}/a_{2n} \rightarrow \delta$  and  $T_{a_{2n+1}}/a_{2n+1} \rightarrow \gamma$ .

But we have already seen that  $(S_n/n - T_n)/n$  converges to zero  $P$ -almost surely, and hence also  $P''$ -almost surely. Thus  $P''$ -almost surely  $S_{a_{2n}}/a_{2n} \rightarrow \delta$  and  $S_{a_{2n+1}}/a_{2n+1} \rightarrow \gamma$ . Since  $\delta > \gamma$ , our desired divergence result follows.  $\square$

*Proof of Theorem 1.* Choose any  $a_1 \in A$ . By Lemma 6, we have an extension  $P'$  of  $P$  such that  $P'$ -almost surely  $S_n/n$  converges to  $a_1$ . Moreover, since  $A$  is a proper non-empty subset of  $[E_P[(X_1)_*], E_P[X_1^*]]$ , the latter interval must contain at least two points and hence  $E_P[(X_1)_*] < E_P[X_1^*]$ , so by the same lemma there is an extension  $P''$  of  $P$  such that  $P''$ -almost surely  $S_n/n$  diverges. Now choose any  $a_2 \in [E_P[(X_1)_*], E_P[X_1^*]] - A$ . Again, by the same lemma there is an extension  $P'''$  of  $P$  such that  $P'''$ -almost surely  $S_n/n$  converges to  $a_2$ .

All the events described in (i)-(v) will happen whenever  $S_n/n \rightarrow a_1$ , and so they all have  $P'$ -measure 1. Events (i), (ii), (iii) and (v) cannot happen when  $S_n/n \rightarrow a_2$ , and so they all have  $P'''$ -measure 0. And event (iv) as  $P''$ -measure 0. Thus, each event has measure 1 under one extension of  $P$  and measure 0 under some other extension, and so by Lemma 5, each event is maximally  $P$ -nonmeasurable.  $\square$

*Proof of Theorem 2.* Let  $a$  and  $\varepsilon$  be as in the statement of the theorem. The conditions of the theorem guarantee that there is a point  $b \in [E_*[X_1], E^*[X_1]] - [a - \varepsilon, a + \varepsilon]$ . Let  $\gamma = |b - a| - \varepsilon$ . Since  $b \notin [a - \varepsilon, a + \varepsilon]$ , we have  $\gamma > 0$ .

By Lemma 6, let  $P'$  be an extension of  $P$  such that  $P'$ -almost surely  $S_n/n$  converges to  $a$ , and let  $P''$  be an extension of  $P$  such that  $P''$ -almost surely



$S_n/n$  converges to  $b$ . Then  $S_n/n$  converges to  $a$  in  $P'$ -probability and to  $b$  in  $P''$ -probability. Hence  $\lim_{n \rightarrow \infty} P'(|S_n/n - a| > \varepsilon) = 0$ . By Lemma 5, we have  $\lim_{n \rightarrow \infty} P_*(|S_n/n - a| > \varepsilon) = 0$ . Moreover,  $0 = \lim_{n \rightarrow \infty} P''(|S_n/n - b| > \gamma) \geq \limsup_{n \rightarrow \infty} P''(|S_n/n - a| \leq \varepsilon)$  by choice of  $\gamma$ . Thus  $P''(|S_n/n - a| \leq \varepsilon)$  converges to 0, and so  $P''(|S_n/n - a| > \varepsilon)$  converges to 1, so that  $P^*(|S_n/n - a| > \varepsilon)$  also converges to 1.  $\square$

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